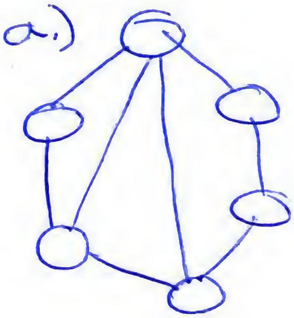
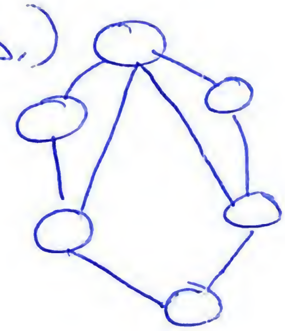
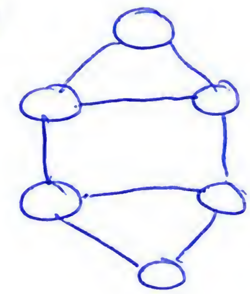
1. a.)



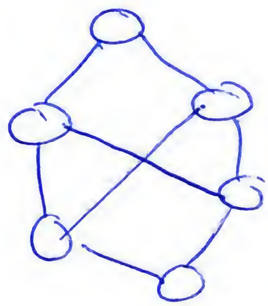
b.)



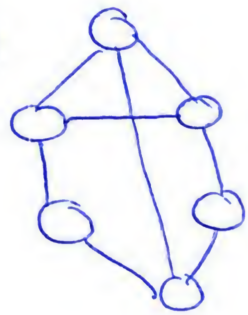
c.)



d.)



e.)



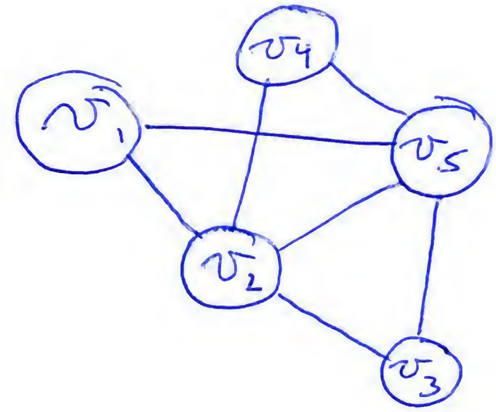
degree sequences	cycle enumeration
a.) $\{4, 3, 3, 2, 2, 2\}$	a.) $\{6, 5, 4, 4, 3, 3\}$
b.) $\{4, 3, 3, 2, 2, 2\}$	b.) $\{6, 5, 5, 4, 3, 3\}$
c.) $\{3, 3, 3, 3, 2, 2\}$	c.) $\{6, 5, 5, 4, 3, 3\}$
d.) $\{3, 3, 3, 3, 2, 2\}$	d.) $\{6, 6, 4, 4, 4, 4\}$
e.) $\{3, 3, 3, 3, 2, 2\}$	e.) $\{6, 5, 4, 4, 3\}$

To show pairwise non-isomorphism, I enumerated all possible cycle lengths in each graph. Only b-c have the same cycle lengths, but they have a differing degree sequence. Isomorphic graphs will have the same subgraphs and degrees.

2.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$G =$



$S = \{2, 4, 2, 2, 4\}$

- By the above sequence, we see the graph has even $d(v)$ $\forall v \in V(G)$. We also see that G is connected. $\therefore$ G is Eulerian.
- G is not bipartite because it contains an odd cycle e.g., $\{v_4, v_2, v_5, v_4\}$.
- G has a single connected component.

3. $G=(V,E)$, $|V| \le |E| \overset{?}{\Rightarrow} \exists C_m \in G$

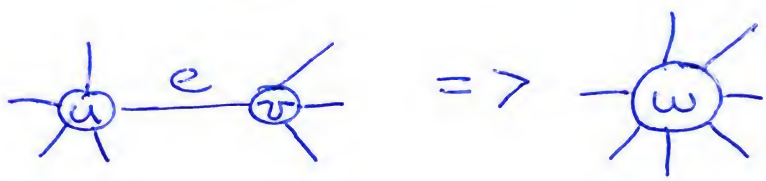
Base case: 

$|V|=1$, $|E|=1$, $C_1 \in G$

Hypothesis: Assume for some $P(k)=H$ where $k=|E(H)| \ge |V(H)| \Rightarrow C_m \in H$

Inductive step: Consider graph G where $P(n)=G$, $k<n=|E(G)|$
$|E(G)| \ge |V(G)|$

Consider some $e \in E(G)$, we perform an edge contraction on e



This produces some graph H where $|E(H)| = |E(G)| - 1$ and $|V(H)| = |E(G)| - 1$.

We invoke our I.H. on H, and assume $\exists C_m \in H$. Edge contraction preserves cycles during re-expansion, so this cycle will also exist on G. $\square$

4. G is simple, connected, $|E(G)| = \text{even}$

$\Rightarrow \exists D = \{P_2, P_2, \ldots, P_2\}$, $P_2 = $ o—o—o

Base Case: o—o—o $\Rightarrow$ decomposition is graph itself

Hypothesis: assume for some $H = P(k)$ where $|E| = k = \text{even}$, H connected, simple $\Rightarrow \exists D = \{P_2, \ldots\}$

Inductive step: Consider G, $P(n) = G$, $k < n = |E(G)|$

We select some arbitrary $P_2 \in G$

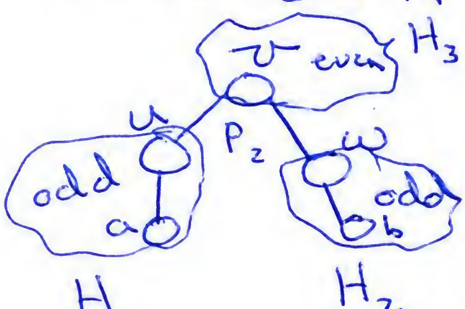
Case 1: $H = G - P_2$, H is connected

We invoke I.H. on H, $D_G = D_H + P_2$

Case 2: $H = G - P_2$, H is disconnected and contains all even components $H_1, H_2, (H_3)$

We invoke I.H. on components, $D_G = D_{H_1} + D_{H_2} + P_2$ $(+ D_{H_3})$

Case 3: $H = G - P_2$, H has two odd components



By selecting an additional edge from each odd component we make them even and can then invoke our I.H. on them.

$$D_G = D_{H_1 - (u,a)} + D_{H_2 - (w,b)} + \{a, u, v\} + \{v, w, b\}$$

$(+ D_{H_3})$ $\square$

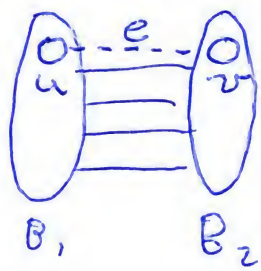
Note: removing two edges splits the graph into at most three components

5. $C_n \notin G$ where n = odd => G is bipartite

Base Case: 

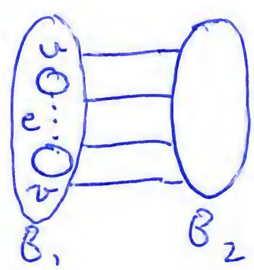
Hypothesis: Assume for $P(k) = H$ where $C_n \in H$, n = odd => H is bipartite

Inductive step: Consider $G = H + e$, $e = (u, v)$

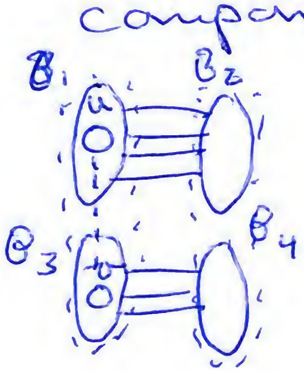
Case 1: 

edge e is added to H between vertices existing in H's bipartite sets B_1, B_2

Bipartition on G = B_1, B_2

Case 2: 

e is added between vertices in B_1 or B_2. In order for the addition of this edge to create a cycle on ~~B~~ G, there must exist some u,v-path $\in H$. As $u, v \in B_1$, this path would necessarily be ~~odd~~ even, creating an odd cycle on G which contradicts our assumption. ∴ no such path exists and u, v are in separate components of H.



We define u's component as B_1, B_2 and v's as B_3, B_4. When adding in e, we then can define G's bipartition as $\{B_1 + B_4\}$ $\{B_2 + B_3\}$